

IV. *On Infinite Series.* By Edward Waring, M. D. F. R. S.
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1. **I**N the Paper, which the Royal Society did me the honour to print, on Summation of Series, is given a method of finding the sum of a series, whose general term $\left(\frac{P}{Q}\right)$ (where $\frac{P}{Q}$ is a fraction reduced to its lowest terms) is a determinate algebraical function of the quantity (z) the distance from the first term of the series, which always terminates when the sum of the series can be expressed in finite terms.

2. The terms of every infinite series must necessarily be given by a function of z , or by quantities which can be reduced to a function of z .

3. Let $Q = A \times A' \times A'^2 \times \dots A'^n \times B \times B' \times B'^2 \times \dots B'^m \times C \times C'^2 \times \dots C'^r \times \&c.$ where $A', A'^2, A'^3 \dots A'^n$, are successive values of A ; that is, result from A by writing in it for z respectively $z+1, z+2, z+3, \dots z+n$; and B', B'^2, B'^3, B'^m , result from B , by writing in it for z respectively $z+1, z+2, z+3, \dots z+m$; but B is not a successive value of A ; &c. Let the numerator $P = E \cdot E' \cdot E'^2 \dots E'^{b-1} \cdot F \cdot F' \cdot F'^2 \dots F'^{k-1} \times L$; $E', E'^2 \dots E'^{b-1}$; $F', F'^2 \dots F'^{k-1}$, &c. denoting successive values of the quantities E, F , &c. respectively; and L , admitting of no divisor of the formula $K \times K'$, where K' is a successive

cessive value of K ; let $L = A \times B \times C \times \&c. \times E^{1b} \times F^{1k} \times \&c. \times p' \times q' \times r' \times \&c. - A^{1n} \times B^{1m} \times C^{1r} \times \&c. \times E \times F \times \&c. \times p \times q \times r \times \&c.$ where $p', q', r', \&c.$ are irrational quantities and successive values of $p, q, r, \&c.$ The factors $A, B, C, \&c. E^{1b}, F^{1k}, \&c.$ being given, the factors $p', q', r', \&c.$ into which they are multiplied in the quantity L will easily be deduced by deducting the preceding irrational factors contained in $A, B, C, \&c. E^{1b}, F^{1k}, \&c.$ from the correspondent irrational factors contained in L ; and in the same manner, from the factors $A^{1n}, B^{1m}, \&c. E, F, \&c.$ can be deduced the irrational factors of the preceding $p, q, r, \&c.$

Assume for the sum of the series sought the quantity

$$E \times E' \times E^{12} \dots E^{1b-1} \times F \times F' \dots F^{1k-1} \times \&c.$$

$A \cdot A' \cdot A^{12} \dots A^{1n-1} \times B \times B' \times B^{12} \dots B^{1m-1} \times C \times C' \dots C^{1r-1} \times \&c. \times p \times q \times r \times s \times \&c. (\alpha z^{m'} + \beta z^{m'-1} + \gamma z^{m'-2} + \&c.) = V$; where m' is a whole number, and $\alpha, \beta, \gamma, \&c.$ are co-efficients to be investigated; write in V for z its successive value $z + 1$, and let the result be W ; reduce the difference $W - V$ into a fraction in its lowest terms, and make the co-efficients of the correspondent terms of the resulting fraction $= (W - V)$ and of the given fraction equal to each other, if possible; and thence may be deduced the sum of the series required.

4. This series will terminate if the sum sought can be expressed by a finite determinate function of z ; if not, it will proceed *in infinitum*, and may be expressed either by a series ascending or descending according to the dimensions of z .

5. If any factor, A or B , or $C, \&c.$ have no successive one in the denominator; or if the greatest dimensions of z in the denominator be greater than its greatest dimensions in the numerator by 1, then the sum of the series is not a finite algebraical function of z .

6. If

6. If in the denominator are deficient some intermediate successive factors, multiply both the numerator and denominator by those deficient factors, and they are supplied: for example, let $A \times A''' \times A'''' \times \&c.$ be factors of the denominator, in which are deficient the factors A', A'', A'''' , &c. multiply both numerator and denominator by the content $A' \times A'' \times A'''' \times \&c.$ and they are restored.

7. If in the denominator is contained the content $A^l \times A^{l'} \times A^{l''} \dots A^{l^n} = H$, in which λ is the least of the indices $l, l', l'', \&c.$: assume $A^\lambda \times A^{\lambda'} \times A^{\lambda''} \dots A^{l^n} \times \alpha = H$, and in the same manner reduce the factors in α ; then assume for the denominator $A^\lambda \times A^{\lambda'} \times A^{\lambda''} \dots A^{n-1} \times \&c.$: for example, let the contents be $A^3 \times A'^5 \times A''^2 \times A'''^5 \times \&c. = H$, then is 2 the least index, and consequently the content reduced as before taught will be $A^2 \cdot A'^2 \cdot A''^2 \cdot A'''^2 \times \alpha = A^2 \cdot A'^2 \cdot A''^2 \cdot A'''^2 \times A \cdot A' \cdot A'' \times A'^2 \cdot A''^2$; but between the factors A' and A''' is deficient the factor A'' ; and between the factors A'^2 and A'''^2 is deficient the factor A''^2 ; multiply the numerator and denominator of the given fraction by the deficient factors $A'' \times A''^2$; assume for the denominator $A^2 \cdot A'^2 \cdot A''^2 \times A \cdot A' \cdot A'' \times A'^2 \cdot A''^2 \times \&c. = A^3 \cdot A'^5 \cdot A''^5 \times \&c. \&c.$

8. If the greatest index of the content H is contained in one factor only, then the sum of the series cannot be expressed in finite terms of the quantity z .

9. The same may be applied to the contents of the several successive values of the quantities $B, C, \&c.$ in the denominator: for example, let the general term be $\frac{1}{1 \cdot 2 \cdot 3 \dots z-2 \times z}$; multiply it into $z-1$ to compleat the deficient term, and it results

$\frac{z-1}{1.2.3\dots z}$; assume, by the preceding method for the sum of the series the quantity $\frac{1}{1.2.3\dots z-1}$, of which the successive term is $\frac{1}{1.2.3\dots z}$, and their difference $\frac{1}{1.2\dots z-1} - \frac{1}{1.2\dots z} = \frac{1}{1.2\dots z-2 \times z}$ the given term.

2. Let the term be Ne^z and e less than 1, which is the term of a geometrical series; then will the sum of the infinite series be $\frac{N}{1-e} \times e^z$, beginning from the term whose distance from the first is z ; for the difference between the two successive sums = $\frac{N}{1-e} (e^z - e^{z+1}) = Ne^z$ the given term.

3. Let the general term be $\frac{(z+n+1-e^n \times z+1)^a}{z+1. z+n+1} \times e^{z+1}$; assume for the sum of the series the subsequent quantity $(z+1. z+2. z+3 \dots z+n)^{-1} \times e^z \times (\alpha + \beta z + \gamma z^2 \dots z^{n-1})$ and by the preceding method the co-efficients α, β, γ , &c. may be found: the sum is known to be $= \frac{a}{z+1} \times e^{z+1} + \frac{a}{z+2} \times e^{z+2} + \frac{a}{z+3} \times e^{z+3} \dots \frac{a}{z+n} e^{z+n}$, which can easily be reduced to the preceding formula.

If the general term be $\frac{T'-T}{T \times T'}$ or $\frac{PQ'-QP'}{QQ'}$; where T and T' , P and P' , Q and Q' , are successive terms; then will the sums of the serieses be $\frac{1}{T}$ or $\frac{P}{Q}$ properly corrected.

10. If the function expressing the general term contain in the denominator a factor or factors, which have no successive one; reduce the factor or factors into an infinite series proceeding according to the dimensions of z , and thence, by the method before given, find the sum of the series. The same method

may be pursued, when the denominator of a fluxion, which is a function of x multiplied into \dot{x} contains the simple power only of a factor or factors; reduce the factor or factors into an infinite series, proceeding according to the dimensions of x , and by the known methods find the fluent of the fluxion.

11. The fluent of the fluxion or sum of the series may be deduced also from the subsequent propositions, from which may be investigated many serieses, whose sums are known.

1. Let $p\dot{P} = Q\dot{q}$, $q\dot{Q} = R\dot{r}$, $r\dot{R} = S\dot{s}$, $s\dot{S} = T\dot{t}$, &c.; then $\int P\dot{p} = Pp - Qq + Rr - Ss + Tt - \&c.$ if only the series converges.

2. Let $pP' + p'P' = Qq'$, $qQ' + q'Q' = Rr'$, $rR' + r'R' = Ss'$, $sS' + s'S' = Tt'$, &c. where P' , p' , Q' , q' , &c. denote the increments of the quantities P , p , Q , q , &c. respectively, then will the integral of the increment $(Pp') = Pp - Qq + Rr - Ss + \&c.$ if only the series converges.

Ex. 1. $\int x^{-n}\dot{x} = \int \frac{x^m \dot{x}}{x^{n+m}} = \frac{1}{x^{n-1}} \left(\frac{1}{m+1} + \frac{n+m}{r'+1} A + \frac{n+r'}{s'+1} B + \frac{n+s'}{t'+1} C + \frac{n+t'}{T+1} D + \&c. \right) = \frac{1}{1-nx^{n-1}}$; whence $\frac{1}{1-n} = \frac{1}{m+1} + \frac{n+m}{r'+1} \times \frac{1}{m+1} \times \frac{n+r'}{s'+1} \times \frac{n+m}{r'+1} \times \frac{1}{m+1} + \&c.$ In this example $\dot{p} = x^m \dot{x}$, $P = x^{-n-m}$, $\dot{q} = x^{r'} \dot{x}$, $Q = x^{-n-r'}$, $\dot{r} = x^{s'} \dot{x}$, &c.

Ex. 2. $\int \frac{x^m \dot{x}}{x^{n+r}} = \int x^{m-n-r} \dot{x} = \frac{1}{m-n-r+1} x^{m-n-r+1} = x^{m-n-r+1} \left(\frac{1}{m+1} + \frac{1}{m+1} \times \frac{n+r}{m+2} + \frac{1}{m+1} \cdot \frac{n+r}{m+2} \cdot \frac{n+r+1}{m+3} + \&c. \right)$; whence $\frac{1}{m-n-r+1} = \frac{1}{m+1} + \frac{n+r}{m+2} A + \frac{n+r+1}{m+3} B + \&c.$ In both these examples the letters A , B , C , &c. denote the preceding terms.

Ex. 3. $\int \frac{x^m \dot{x}}{1+x} = \frac{x^{m+1}}{1+x} \left(\frac{1}{m+1} + \frac{1}{m+1 \cdot m+2} \times \frac{x}{1+x} + \frac{2}{m+1 \cdot m+2 \cdot m+3} \times \frac{x^2}{(1+x)^2} + \frac{2 \cdot 3}{m+1 \cdot m+2 \cdot m+3 \cdot m+4} \cdot \frac{x^3}{(1+x)^3} + \&c. \right)$

Ex.

$$\text{Ex. 4.} \int \frac{a'x^m}{(a+bx^n+cx^{2n})^b} = \frac{1}{m+1} \frac{a'x^{m+1}}{(a+bx^n+cx^{2n})^b} + \frac{a'x^{m+1}}{m+1 \cdot (a+bx^n+cx^{2n})^{b+1}} \\ \times \left(\frac{nbbx^{n-1}}{m+n+1} + \frac{2nbcx^{2n-1}}{m+2n+1} \right) + \&c.$$

12. Let the general term of an infinite series be

$$\frac{A}{z+\alpha \cdot z+\alpha+1 \cdot z+\alpha+2 \dots z+\alpha+n \times z+\beta \cdot z+\beta+1 \cdot z+\beta+2 \dots z+\beta+r \\ m \times z+\gamma \cdot z+\gamma+1 \cdot z+\gamma+2 \dots z+\gamma+r \times z+\delta \cdot z+\delta+1 \cdot z+\delta+2 \dots \\ z+\delta+s \times \&c.}; \text{ where } \alpha-\beta, \alpha-\gamma, \beta-\gamma, \&c. \text{ are not whole} \\ \text{numbers; the sum of the series can always be expressed} \\ \text{in finite terms; if the sum of the fractions } \frac{1}{1 \cdot 2 \cdot 3 \dots n} \times \\ \frac{1}{\beta-\alpha \cdot \beta-\alpha+1 \cdot \beta-\alpha+2 \dots \beta-\alpha+m \times \gamma-\alpha \cdot \gamma-\alpha+1 \cdot \gamma-\alpha+2 \dots \gamma-\alpha+r \\ \times \delta-\alpha \cdot \delta-\alpha+1 \cdot \delta-\alpha+2 \dots \delta-\alpha+s \times \&c.} - \frac{1}{1 \times 1 \cdot 2 \cdot 3 \dots n-1} \times \\ \frac{1}{\beta-\alpha-1 \cdot \beta-\alpha \cdot \beta-\alpha+1 \cdot \beta-\alpha+2 \dots \beta-\alpha+m-1 \times \gamma-\alpha-1 \cdot \gamma-\alpha \cdot \gamma-\alpha+1 \\ \dots \gamma-\alpha+r-1 \times \delta-\alpha-1 \cdot \delta-\alpha \cdot \delta-\alpha+1 \dots \delta-\alpha+s-1 \times \&c.} + \\ \frac{1}{1 \cdot 2 \times 1 \cdot 2 \cdot 3 \dots n-2} \times \frac{1}{\beta-\alpha-2 \cdot \beta-\alpha-1 \cdot \beta-\alpha \cdot \beta-\alpha+1 \dots \beta-\alpha+n-2 \times \gamma-\alpha \\ -2 \cdot \gamma-\alpha-1 \cdot \gamma-\alpha \dots \gamma-\alpha+n-2 \times \delta-\alpha-2 \cdot \delta-\alpha-1 \cdot \delta-\alpha \dots \delta-\alpha+n-2 \times \&c.} \\ - \frac{1}{1 \cdot 2 \cdot 3 \times 1 \cdot 2 \dots n-3} \times \frac{1}{\beta-\alpha-3 \cdot \beta-\alpha-2 \dots \beta-\alpha+n-3 \times \gamma-\alpha-3 \cdot \gamma-\alpha-2 \dots \gamma-\alpha+n-3 \times \delta-\alpha-3 \cdot \delta-\alpha-2 \dots \delta-\alpha+s-3 \times \&c.} + \dots \frac{1}{1 \cdot 2 \cdot 3 \dots n} \times \\ \frac{1}{\beta-\alpha-n \cdot \beta-\alpha-n+1 \dots \beta-\alpha-n+m \times \gamma-\alpha-n \cdot \gamma-\alpha-n+1 \dots \gamma-\alpha-n+r \times \\ \delta-\alpha-n \cdot \delta-\alpha-n+1 \dots \delta-\alpha-n+s \times \&c.} = 0; \frac{1}{1 \cdot 2 \cdot 3 \dots m} \times \frac{1}{\alpha-\beta \cdot \alpha-\beta+1 \\ 1 \cdot \alpha-\beta+2 \dots \alpha-\beta+m \times \gamma-\beta \cdot \gamma-\beta+1 \dots \gamma-\beta+r \times \delta-\beta \cdot \delta-\beta+1 \\ \dots \delta-\beta+2 \dots \delta-\beta+s \times \&c.} - \frac{1}{1 \times 1 \cdot 2 \cdot 3 \dots m-1} \times \frac{1}{\alpha-\beta-1 \cdot \alpha-\beta \dots \alpha-\beta+n-1}$$

$$\begin{aligned}
 & \frac{1}{n-1 \times \gamma-\beta-1 \cdot \gamma-\beta_1 \cdot \gamma-\beta+r-1 \times \delta-\beta-1 \cdot \delta-\beta \cdot \delta-\beta+s-1 \times \&c.} + \\
 & \frac{1}{1 \cdot 2 \times 1 \cdot 2 \cdot 3 \dots m-2} \times \frac{1}{\alpha-\beta-2 \cdot \alpha-\beta-1 \cdot \alpha-\beta \cdot \alpha-\beta+n-2 \times \gamma-\beta-2} \\
 & \frac{1}{\gamma-\beta-1 \dots \gamma-\beta+r-2 \times \delta-\beta-2 \cdot \delta-\beta-1 \dots \delta-\beta+s-2 \times \&c.} \\
 & \frac{1}{1 \cdot 2 \cdot 3 \times 1 \cdot 2 \cdot 3 \dots m-3} \times \frac{1}{\alpha-\beta-3 \times \alpha-\beta-2 \times \alpha-\beta-1 \dots \alpha-\beta+n-3 \times \gamma-} \\
 & \frac{1}{\beta-3 \cdot \gamma-\beta-2 \cdot \gamma-\beta-1 \dots \gamma-\beta+r-3 \times \delta-\beta-3 \cdot \delta-\beta+s-3 \times \&c.} \dots \\
 & \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \dots m} \times \frac{1}{\alpha-\beta-m \cdot \alpha-\beta-m+1 \dots \alpha-\beta+n-m \times \gamma-\beta-m \cdot \gamma-\beta-} \\
 & \frac{1}{m+1 \dots \gamma-\beta-m+r \times \delta-\beta-m \cdot \delta-\beta-m+1 \dots \delta-\beta-m+s \times \&c.} = 0; \\
 & \frac{1}{1 \cdot 2 \cdot 3 \dots r} \times \frac{1}{\alpha-\gamma \cdot \alpha-\gamma+1 \dots \alpha-\gamma+n \times \beta-\gamma \cdot \beta-\gamma+1 \cdot \beta-\gamma+m} \\
 & \frac{1}{\times \delta-\gamma \cdot \delta-\gamma+1 \dots \delta-\gamma+s \times \&c.} - \&c. = 0, \&c. = 0; \text{ or, to explain}
 \end{aligned}$$

it otherwise, assume the quantities $\alpha, \alpha+1, \alpha+2, \alpha+3, \dots \alpha+n$; $\beta, \beta+1, \beta+2, \dots \beta+m$; $\gamma, \gamma+1, \gamma+2, \dots \gamma+r$; $\delta, \delta+1, \delta+2, \dots \delta+s$; &c. subtract one α from all the remaining quantities $\alpha+1, \alpha+2, \dots \alpha+n$; $\beta, \beta+1, \dots \beta+m$; $\gamma, \gamma+1, \dots \gamma+r$; $\delta, \delta+1, \dots \delta+s$; &c. and multiply all the differences resulting $1, 2, 3, \dots n$; $\beta-\alpha, \beta-\alpha+1, \dots \beta-\alpha+m$; $\gamma-\alpha, \gamma-\alpha+1, \dots \gamma-\alpha+r$; $\delta-\alpha, \delta-\alpha+1, \dots \delta-\alpha+s$, &c. into each other, and call the content p^r . In the same manner subtract $\alpha+1$ from all the remaining quantities $\alpha, \alpha+2, \alpha+3, \dots \alpha+n$; $\beta, \beta+1, \dots \beta+m$; $\gamma, \gamma+1, \dots \gamma+r$; $\delta, \delta+1, \dots \delta+s$; &c.; and let the remainders $-1, 1, 2, 3, \dots n-1$; $\beta-\alpha-1, \beta-\alpha, \beta-\alpha+1, \dots \beta-\alpha+m-1$; $\gamma-\alpha-1, \gamma-\alpha, \gamma-\alpha+1, \dots \gamma-\alpha+r-1$; $\delta-\alpha-1, \delta-\alpha, \delta-\alpha+1, \dots \delta-\alpha+s-1$, &c. be multiplied together, and their content be called p^{r+1} . In the same manner subtract $\alpha+2, \alpha+3, \dots \alpha+n$ respectively from all the remaining quantities, and let the differences resulting

suming be multiplied together, and their respective contents be called $p^{\alpha+2}, p^{\alpha+3}, p^{\alpha+4}, \dots p^{\alpha+n}$; then, if the sum of the series can be found, will $\frac{1}{p^{\alpha}} + \frac{1}{p^{\alpha+1}} + \frac{1}{p^{\alpha+2}} + \frac{1}{p^{\alpha+3}} \dots \frac{1}{p^{\alpha+n}} = 0$.

In the same manner subtract $\beta, \beta+1, \beta+2, \dots \beta+m$ respectively from all the remaining quantities, and multiply their respective remainders into each other, and call their contents respectively $p^{\beta}, p^{\beta+1}, p^{\beta+2}, \dots p^{\beta+m}$, then will $\frac{1}{p^{\beta}} + \frac{1}{p^{\beta+1}} + \frac{1}{p^{\beta+2}} \dots \frac{1}{p^{\beta+m}} = 0$.

Subtract $\gamma, \gamma+1, \gamma+2, \dots \gamma+r$ respectively from all the remaining quantities, and multiply their respective remainders into each other, and call their contents respectively $p^{\gamma}, p^{\gamma+1}, p^{\gamma+2}, \dots p^{\gamma+r}$; then will $\frac{1}{p^{\gamma}} + \frac{1}{p^{\gamma+1}} + \frac{1}{p^{\gamma+2}} \dots \frac{1}{p^{\gamma+r}} = 0$.

Subtract $\delta, \delta+1, \delta+2, \dots \delta+s$ from all the remaining quantities, and multiply their respective remainders into each other, and call their contents respectively $p^{\delta}, p^{\delta+1}, p^{\delta+2}, p^{\delta+s}$; then will $\frac{1}{p^{\delta}} + \frac{1}{p^{\delta+1}} + \frac{1}{p^{\delta+2}} + \frac{1}{p^{\delta+s}} = 0$; and so on; and, *vice versa*, if the sum of the above-mentioned fractions be respectively $= 0$; then the sum of the series, whose general term is the given one, can be found; otherwise not.

13. If the sum of the series, whose general term is

$$\frac{1}{z+\alpha \cdot z+\alpha+1 \dots z+\alpha+n \times z+\beta \cdot z+\beta+1 \dots z+\beta+m \times z+\gamma \cdot z+\gamma+1 \dots z+\gamma+r \times z+\delta \cdot z+\delta+1 \dots z+\delta+s \times \&c. = H},$$

can be found in finite terms; then the sum of a series, whose general term is $\frac{az^l + bz^{l-1} + cz^{l-2} + \&c.}{H}$ (where l denotes an affirmative number)

can be also expressed in finite terms.

14. Let

14. Let $A \times \overline{z+\beta} \cdot \overline{z+\gamma} \cdot \overline{z+\delta} \times \overline{z+\varepsilon} \times \&c. + B \times \overline{z+\alpha} \times \overline{z+\gamma} \cdot \overline{z+\delta} \cdot \overline{z+\varepsilon} \cdot \&c. + C \times \overline{z+\alpha} \cdot \overline{z+\beta} \times \overline{z+\delta} \cdot \overline{z+\varepsilon} \cdot \&c. + D \times \overline{z+\alpha} \cdot \overline{z+\beta} \cdot \overline{z+\gamma} \times \overline{z+\varepsilon} \cdot \&c. + E \times \overline{z+\alpha} \cdot \overline{z+\beta} \cdot \overline{z+\gamma} \cdot \overline{z+\delta} \times \&c. = 1$, whatever may be the value of z ; then will

$$A = \frac{1}{\beta-\alpha \cdot \gamma-\alpha \cdot \delta-\alpha \cdot \&c.}, B = \frac{1}{\alpha-\beta \cdot \gamma-\beta \cdot \delta-\beta \cdot \&c.}, C = \frac{1}{\alpha-\gamma \cdot \beta-\gamma \cdot \delta-\gamma \cdot \&c.},$$

$$D = \frac{1}{\alpha-\delta \cdot \beta-\delta \cdot \gamma-\delta \cdot \&c.}, \&c.$$

15. Let the general term of a series be $\frac{1}{H} \times e^z$, where e is less than 1; the sum of the series can be expressed in finite

terms, when the above-mentioned quantities $\frac{1}{p\alpha} + \frac{\frac{1}{e}}{p\alpha+1} + \frac{\frac{1}{e^2}}{p\alpha+2}$

$$+ \frac{\frac{1}{e^3}}{p\alpha+3} \dots + \frac{\frac{1}{e^n}}{p\alpha+n} = 0, \frac{1}{p\beta} + \frac{\frac{1}{e}}{p\beta+1} + \frac{\frac{1}{e^2}}{p\beta+2} + \frac{\frac{1}{e^3}}{p\beta+3} \dots + \frac{\frac{1}{e^m}}{p\beta+m} = 0,$$

$$\frac{1}{p\gamma} + \frac{\frac{1}{e}}{p\gamma+1} + \frac{\frac{1}{e^2}}{p\gamma+2} \dots \frac{\frac{1}{e^r}}{p\gamma+r} = 0, \frac{1}{p\delta} + \frac{\frac{1}{e}}{p\delta+1} + \frac{\frac{1}{e^2}}{p\delta+2} + \frac{\frac{1}{e^3}}{p\delta+3} \dots$$

$$\frac{\frac{1}{e^s}}{p\delta+s} = 0, \&c. = 0. \text{ This never happens unless } e = 1.$$

16. If the general term be any rational function of z into the exponential e^z , viz. $\frac{az^l + bz^{l-1} + cz^{l-2} + \&c.}{z+\alpha \times z+\alpha + b \times z+\alpha + k \times \&c. \times z+\beta^n \times$

$$\frac{\&c.}{z+\beta + b'^{n'} \times z+\beta + k'^{n''} \times \&c. \times z+\gamma \times z+\gamma + b''^{r'} \times z+\gamma + k''^{r''} \times \&c.} \times e^z = K,$$

where $b, b', b'', \&c. k, k', k'', \&c. l, m, m', m'', \&c. n, n', n'', \&c. r, r', r'', \&c. \&c.$ denote whole numbers, and neither

$\alpha - \beta$, nor $\alpha - \gamma$, nor $\beta - \gamma$, &c. are whole numbers: let m be the greatest of the indices $m, m', m'',$ &c.; n the greatest of the indices $n, n', n'',$ &c.; r the greatest of the indices $r, r', r'',$ &c.; then, from the sums of the serieses given, whose general

terms are $\frac{e^z}{(z+\alpha)^m}, \frac{e^z}{(z+\alpha)^{m-1}}, \frac{e^z}{(z+\alpha)^{m-2}}, \dots, \frac{e^z}{(z+\alpha)^2}, \frac{e^z}{z+\alpha},$
 $\frac{e^z}{(z+\beta)^n}, \frac{e^z}{(z+\beta)^{n-1}}, \dots, \frac{e^z}{z+\beta}; \frac{e^z}{(z+\gamma)^r}, \frac{e^z}{(z+\gamma)^{r-1}}, \dots, \frac{e^z}{z+\gamma},$ &c. can

be deduced the sum of the above series, whose general term is given above; multiply each of these terms into unknown coefficients $e', f, g, h,$ &c.; then reduce them to a common denominator, which is the same as the denominator of the given general term, and add them together, and make the correspondent terms of the sum resulting equal to the correspondent terms of the numerator $az^l + bz^{l-1} + cz^{l-2} +$ &c. of the given general term; and from the equations resulting can be deduced the co-efficients $e', f, g, h,$ &c. and thence from the given sums the sum of the series required.

Approximations to the sums of the serieses may be deduced from the methods given in the *Meditationes Analyticae*. The sum of some few cases have been given from the periphery of the circle: for example, when α and m are whole numbers, and $e = 1$; or, more particularly, when $m = 2$ and $e = \frac{1}{2}$, and some other particular cases, which may be with nearly the same facility calculated from approximations; the cases given indeed are so few, unless when $e = 1$, that they can very rarely be applied.

17. If the dimensions of z in the numerator be equal or greater than its dimensions in the denominator; that is, l be equal or greater than $m + m' + m'' +$ &c. $+ n + n' + n'' +$ &c. $+ r + r' + r''$

+ &c. &c. reduce the fractions to a mixed number, so that the dimensions of z in the numerator of the fraction be less than its dimensions in the denominator, and the integral part be $Ae^z + Bze^z + Cz^2e^z + Dz^3e^z + \dots Hze^{1m}e^z$: the sum of the infinite series whose term is $Ae^z + Bze^z + Cz^2e^z + Dz^3e^z + Ez^4e^z + \dots$

$$+ Hze^m e^z = \frac{Ae}{1-e} + B \left(\frac{1}{(1-e)^2} - \frac{1}{1-e} \right) + Ce \left(\frac{1 \cdot 2}{(1-e)^3} - \frac{1}{(1-e)^2} \right) + De \left(\frac{1 \cdot 2 \cdot 3}{(1-e)^4} - \frac{1+2 \times 1 \cdot 2}{(1-e)^3} + \frac{1}{(1-e)^2} \right) + Ee \left(\frac{1 \cdot 2 \cdot 3 \cdot 4}{(1-e)^5} - \&c. \right): \text{the sum}$$

of a series (whose general term is $z^m e^z$) = $\frac{1 \cdot 2 \cdot 3 \cdot 4 \dots m}{(1-e)^{m+1}} e -$

$$\frac{{}^1S^{m-1}}{(1-e)^m} 1 \cdot 2 \cdot 3 \dots m-1 e + \frac{{}^1S^{m-1} \times {}^1S^{m-2} - {}^2S^{m-1}}{(1-e)^{m-1}} \times 1 \cdot 2 \cdot 3 \dots m-2 \cdot e$$

$$+ \dots + \frac{L}{(1-e)^{m-b}} \times 1 \cdot 2 \cdot 3 \dots m-b-1 \dots \pm \frac{1}{(1-e)^2}, \text{ where } L \text{ is}$$

equal to the sum of all quantities of the following sort, ${}^a S^{m-1} \times {}^\beta S^{m-a-1} \times {}^\gamma S^{m-a-\beta-1} \times {}^\delta S^{m-a-\beta-\gamma-1} \times {}^\epsilon S^{m-a-\beta-\gamma-\delta-1} \times \&c.$ where $\alpha, \beta, \gamma, \delta, \&c.$ are whole affirmative numbers; (in the preceding notation by ${}^\pi S$ is designed the sum of the contents of every π of the following numbers $1, 2, 3, 4, 5, \dots \rho$); and $\alpha + \beta + \gamma + \delta + \epsilon + \&c. = b + 1$; the above-mentioned product ${}^a S^{m-1} \times {}^\beta S^{m-a-1} \times \&c.$ is to be taken affirmative or negative, according as the number of letters $\alpha, \beta, \gamma, \delta, \&c.$ is even or uneven.

The sum of the series $z^m \times e^z$ may also be found by assuming for it $(az^m + bz^{m-1} + cz^{m-2} + dz^{m-3} \dots k) e^z$; then, finding its successive term $(a \times z + 1 e + b \times z + 1 e + c \times z + 1 e + \&c.) e^z$, and taking the difference between it and the assumed quantity, there results $(a \times e - 1 z^m + m \cdot e + b e - 1 z^{m-1} + \&c.) e^z$; by equating it to the given term $z^m e^z$ are deduced the subsequent equations $a \times e - 1 = 1, mae + be - 1 = 0, \&c.$ whence

$$a = \frac{1}{e-1}, \quad b = \frac{1}{e} - ma, \quad \&c.$$

If $e = 1$, then assume $az^{m+1} + bz^m + \&c.$ for the sum sought, which rule was first taught by M. J. BERNOULLI.

If the term be $z^m f^{b^2}$; for f^b substitute e , and there results $z^m e^m$ the same as before.

18. Let $P = A + Bx^n + Cx^{2n} + Dx^{3n} + \&c.$ then will the sum of the series

$$\frac{A}{\alpha \cdot \beta \cdot \gamma \cdot \delta \cdot \&c.} + \frac{Bx^n}{\alpha+n \cdot \beta+n \cdot \gamma+n \cdot \&c.} + \frac{Cx^{2n}}{\alpha+2n \cdot \beta+2n \cdot \gamma+2n \cdot \&c.} + \&c. = \frac{1}{\alpha \beta \gamma \delta \cdot \&c.} \times P - \frac{1}{\alpha} \cdot \frac{1}{\beta-\alpha} \cdot \frac{1}{\gamma-\alpha} \cdot \frac{1}{\delta-\alpha} \cdot \&c. \times \&c. x^{-\alpha} \int x^{\alpha} \dot{p} - \frac{1}{\beta} \cdot \frac{1}{\alpha-\beta} \cdot \frac{1}{\gamma-\beta} \cdot \frac{1}{\delta-\beta} \cdot \&c. \times x^{-\beta} \int x^{\beta} \dot{p} - \frac{1}{\gamma} \cdot \frac{1}{\alpha-\gamma} \cdot \frac{1}{\beta-\gamma} \cdot \frac{1}{\delta-\gamma} \cdot \&c. \times x^{-\gamma} \int x^{\gamma} \dot{p} - \&c.$$

This may be proved from the subsequent arithmetical proposition

$$\frac{1}{\alpha} \cdot \frac{1}{\beta-\alpha} \cdot \frac{1}{\gamma-\alpha} \cdot \frac{1}{\delta-\alpha} \cdot \&c. + \frac{1}{\beta} \cdot \frac{1}{\alpha-\beta} \cdot \frac{1}{\gamma-\beta} \cdot \frac{1}{\delta-\beta} \cdot \&c. + \frac{1}{\gamma} \cdot \frac{1}{\alpha-\gamma} \cdot \frac{1}{\beta-\gamma} \cdot \frac{1}{\delta-\gamma} \cdot \&c. + \frac{1}{\delta} \cdot \frac{1}{\alpha-\delta} \cdot \frac{1}{\beta-\delta} \cdot \frac{1}{\gamma-\delta} \cdot \&c. = \frac{1}{\alpha \cdot \beta \cdot \gamma \cdot \delta \cdot \&c.}.$$

19. Let the general term of the above-mentioned series $A + Bx^n + Cx^{2n} + \&c.$ be Hx^{ln} ; then from the sums of the series p , and the fluents of the fluxions $x^{\alpha} \dot{p}$, $x^{\beta} \dot{p}$, $x^{\gamma} \dot{p}$, $x^{\delta} \dot{p}$, $\&c.$ being given there follows the sum of a series, whose general term is

$$\frac{az^{ln} + bz^{l'n-1} + cz^{l'n-2} + \&c.}{\alpha+nz \cdot \beta+nz \cdot \gamma+nz \cdot \delta+nz \cdot \&c.} \times Hx^{ln},$$

where l denotes a whole number.

If $H = m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \dots \cdot \frac{m-z}{z+1}$, or $= m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \dots \cdot \frac{m-lz}{lz+1}$, where l is a whole number, and $m, \alpha', \beta', \gamma', \&c.$ are either whole numbers or fractions whose denominator is 2, and $\alpha = \alpha'n, \beta = \beta'n, \&c.$ the sum of the above-mentioned series can be found by finite terms, circular arcs and logarithms. If

If $H = \frac{1}{1 \cdot 2 \cdot 3 \dots z}$ or $= \frac{1}{1 \cdot 2 \cdot 3 \dots lz}$, and l, α, β, γ , &c. whole numbers and $n = 1$; then can the sum of any series of the above-mentioned formula be found in finite algebraical functions of x , and the circular arcs and logarithms of them.

P A R T II.

1. THE doctrine of proportional parts was probably very early known in the æra of science; for when men could not find the exact value of a quantity, they were induced to find near approximations by trials, and from thence, by proportion, an approximation still nearer: which method is commonly denominated the Rule of False.

This was often found to deviate considerably from the exact value; and the same operation was repeated, which frequently produced a nearer approximate value, and so on.

This method of approximations, the most general yet known, has been used in resolving problems by several of the most eminent mathematicians in different ages, and in this particularly by M. EULER.

2. The following observation, I believe, was first published in the *Meditationes*, in the year 1770, viz. that the convergency of the approximate values, found by the rule of false and method of infinite series, generally depended on this, viz. how

how much nearer the approximate assumed is to one value of the quantity sought possible or impossible than to any other, and not to the quantity itself: hence, when two or more (n) values of the quantity sought are nearly equal, it is necessary to recur to more difficult rules, *viz.* to three or more trials; as, for example, let two roots be nearly equal, and write a , $a + \pi$, and $a + \rho$, for the unknown quantity in the given equation made $= 0$, and let the quantities resulting be A , B , and C , then will more near approximations to the two roots nearly equal of the given equation be $a +$ the two roots (x) of the quadratic $\left(\frac{A}{\pi\rho} + \frac{B}{\pi(\pi-\rho)} + \frac{C}{\rho(\rho-\pi)}\right)x^2 - \left(A \times \frac{\pi+\rho}{\pi\rho} + B \times \frac{\rho}{\pi \cdot (\pi-\rho)} + \frac{C\pi}{\rho \cdot (\rho-\pi)}\right)x + A = 0$: for write a , π , and ρ , respectively for x in the equation, and there will result the quantities A , B , and C .

More generally, substitute for x in the given equation the quantities a , $a + \pi$, $a + \rho$, $a + \sigma$, $a + \tau$, $a + \nu$, &c. where π , ρ , σ , τ , &c. are very small quantities; and let the quantities resulting be A , B , C , D , E , F , &c.; then will more near approximations to the (n) roots of the given equation be $a + \alpha$, $a + \beta$, $a + \gamma$, $a + \delta$, &c. where α , β , γ , δ , &c. are the n roots (e) of the given equation
$$\frac{(e-\pi)(e-\rho)(e-\sigma)(e-\tau)\&c.}{\pm \pi \rho \sigma \tau, \&c.} \times A + \frac{e(e-\rho)(e-\sigma)(e-\tau)\&c.}{\pi(\pi-\rho)(\pi-\sigma)(\pi-\tau)\&c.} \times B + \frac{e(e-\pi)(e-\sigma)(e-\tau)\&c.}{\rho(e-\pi)(\rho-\sigma)(\rho-\tau)\&c.} \times C + \&c. = A - \left(\frac{A}{\pi} - \frac{B}{\pi}\right)e + \left(\frac{A}{\pi\rho} + \frac{B}{\rho(\pi-\rho)} + \frac{C}{\rho(\rho-\pi)}\right) \times e \cdot \overline{e-\pi} - \left(\frac{A}{\pi\rho\sigma} - \frac{B}{\pi \cdot \pi-\rho}{\pi-\sigma} - \frac{C}{\rho(\rho-\pi)(\rho-\sigma)} - \frac{D}{\sigma(\sigma-\pi)(\sigma-\rho)}\right) \times e \cdot \overline{e-\pi} \cdot \overline{e-\rho} + \left(\frac{A}{\pi\rho\sigma\tau} + \frac{B}{\pi(\pi-\rho)(\pi-\sigma)(\pi-\tau)} + \frac{C}{\rho(e-\pi)(\rho-\sigma)(\rho-\tau)} + \frac{D}{\sigma(e-\pi)(\sigma-\rho)(\sigma-\tau)} + \frac{E}{\tau(\tau-\pi)(\tau-\rho)(\tau-\sigma)}\right) \cdot \overline{e-\pi} \cdot \overline{e-\rho} \cdot \overline{e-\sigma} - \left(\frac{A}{\pi\rho\sigma\tau\nu} - \frac{B}{\pi(\pi-\rho)(\pi-\sigma)(\pi-\tau)(\pi-\nu)} - \frac{C}{\rho(e-\pi)(\rho-\sigma)(\rho-\tau)}\right) (\&c.)$$

$$\frac{\frac{D}{(\xi-r)(\xi-v)} - \frac{E}{\sigma(\sigma-\pi)(\sigma-\xi)(\sigma-\tau)(\sigma-v)} - \frac{F}{\tau(\tau-\pi)(\tau-\xi)(\tau-\sigma)(\tau-v)}}{\frac{F}{v(v-\pi)(v-\xi)(v-\sigma)(v-\tau)}} e, e-\pi, e-\rho, e-\sigma, e-\tau + \&c. = 0.$$

Their resolutions were first given in the *Meditationes Analyticae*, published in the year 1774, and require the extraction of a quadratic, cubic, and in general of an equation of (n) dimensions; which rules will often give a nearer approximate than the preceding, when the roots are not nearly equal.

3. These rules may be applied to find approximations to the roots of algebraical equations: for example, let the algebraical equation be $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, substitute in it for x two quantities a , and $a + e$ much nearer to one root than to any other, and there result $a^n - pa^{n-1} + qa^{n-2} - \&c. = A$, and $(a + e)^n - p(a + e)^{n-1} + q(a + e)^{n-2} - \&c. = B$; then, by the rule of

false, $B - A : e :: A : \frac{a^n - pa^{n-1} + qa^{n-2} - \&c.}{(na^{n-1} - n - 1pa^{n-2} + n - 2qa^{n-3} - \&c.) + \&c.} = b$;

whence $a - b$ a near approximate value to the root sought. If the quantities, in which are involved $e, e^2, e^3, \&c.$ on account of e being very small, be rejected, then will the approximate sought $b =$

$\frac{a^n - pa^{n-1} + qa^{n-2} - \&c.}{na^{n-1} - n - 1pa^{n-2} + n - 2qa^{n-3} - \&c.}$; which will nearly be the same

as found, where a near approximate is given, from the method given by VIETA, HARRIOT, OUGHTRED, NEWTON, DE LAGNY, HALLEY, &c.

4. From this expression it follows, that if (u) be a root of the equation $na^{n-1} - n - 1pa^{n-2} + \&c. = 0$, of which the roots are limits between the roots of the given equation, the approximation found will be infinite.

5. In finding these approximations, when there are two or more quantities contained in the given equation dependent on each

each other, as the arc and the sine, it is necessary that both should be corrected in every approximation to such a degree as the subsequent approximations require.

6. In the *Meditationes* it is observed, that in any algebraical equation $x^n - ax^{n-1} + bx^{n-2} - cx^{n-3} + dx^{n-4} - ex^{n-5} \dots \pm gx^{n-m+1} \mp bx^{n-m} \pm kx^{n-m-1} \mp lx^{n-m-2} \pm \&c. = 0$, if a be much greater than $\frac{b}{a}$, and $\frac{b}{a}$ has to $\frac{c}{b}$ a much greater ratio than $a : \frac{b}{a}$; and in the same manner $\frac{c}{b}$ has to $\frac{d}{c}$ a much greater ratio than $\frac{b}{a} : \frac{c}{b}$, and so on; then will a be a near approximate to the greatest root of the algebraical equation; $\frac{b}{a}$ a near approximate to the second; $\frac{c}{b}$ a near approximate to the third, and $\frac{k}{b}$ a near approximate to a root, which is much less than m roots of the given equation, but much greater than the remaining $(n - m - 1)$ roots.

If the equation above-mentioned $x^n \pm ax^{n-1} + \&c. = 0$, or which is the same, $1 \pm \frac{a}{x} + \frac{b}{x^2} \pm \&c. = 0$ be infinite; then will, in like manner, all its roots be possible and their approximate values $a, \frac{b}{a}, \frac{c}{b}, \&c.$ as before.

This easily appears by substituting for $a, b, c, \&c.$ their values in terms of the root of the equation.

7. A nearer approximate to the above-mentioned root will be $\frac{k}{b} - \left(\frac{1}{k} - \frac{gk^2}{b^3} \right) + \&c.$

8. Equations, of which the fluxions of the quantities contained in the given equations can be found, may be reduced to infinite algebraical equations, in which is involved no irrational function of the unknown quantities contained in the given equations by the incremental theorem; viz. let $A = 0$ be the given equation,

equation, and (*a*) an approximate much more near to the root required (π) of the given equation than to any other: write *a* for (x) the unknown quantity sought in the subsequent quantities, \dot{A} , \ddot{A} , $\ddot{\ddot{A}}$, &c.; and let there result the correspondent quantities α , β , γ , δ , &c.; then will $\pi - a$ be a root (*e*) of the infinite equation $\alpha \pm \beta e + \frac{1}{1 \cdot 2} \gamma e^2 \pm \frac{1}{1 \cdot 2 \cdot 3} \delta e^3 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \varepsilon e^4 \pm \&c. = 0$, of which a root of the equations $\alpha \pm \beta e = 0$, $\alpha \pm \beta e + \frac{1}{1 \cdot 2} \gamma e^2 = 0$, &c. will be an approximate. If two roots of the given equation are nearly $= a$, then it is necessary to recur to an equation not inferior to a quadratic.

9. The successive approximate values found by these and like rules will ultimately be to each other in a greater than any geometrical ratio: for example, let $\frac{as}{a+s}$ be an approximate to a root of the quadratic $x^2 - (a+s)x + as = 0$, then will the new addition to the approximate to the root *s* found by the common method at the distance $n-1$ from the first approximate be $\frac{s^b}{a^{b-1}}$ nearly, where $b = 2^{n-1}$.

10. Let an equation $x^n - Px^{n-1} + Qx^{n-2} - Rx^{n-3} + Sx^{n-4} - \&c. = 0$, of which the roots are *a*, *b*, *c*, *d*, &c.; and an equation $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$, where *p*, *q*, *r*, *s*, &c. differ from the co-efficients *P*, *Q*, *R*, *S*, &c. by very small quantities: assume the (*n*) equations $\pi + \rho + \sigma + \tau + \&c. = p - P = \alpha$, $a\pi + b\rho + c\sigma + d\tau + \&c. = P\alpha - q + Q = \beta$, $a^2\pi + b^2\rho + c^2\sigma + d^2\tau + \&c. = P\beta - Q\alpha + r - R = \gamma$, $a^3\pi + b^3\rho + c^3\sigma + d^3\tau + \&c. = P\gamma - Q\beta + R\alpha - s + S = \delta$, &c.; and from them find the unknown quantities π , ρ , σ , τ , &c.; then will $a + \pi$, $b + \rho$, $c + \sigma$, $d + \tau$, &c. be nearly the *n* roots of the equation $x^n - px^{n-1} + qx^{n-2}$

$-\&c.=0$, and consequently $\pi, \rho, \&c.$ nearly $-\frac{a^n - pa^{n-1} + qa^{n-2} - \&c.}{na^{n-1} - n-1pa^{n-2} + \&c.}$ $?$
 $-\frac{b^n - pb^{n-1} + qb^{n-2} - \&c.}{nb^{n-1} - n-1pb^{n-2} + \&c.}$, $\&c.$; whence the convergency of the approximate values found by this rule depends on the principle before delivered.

10. Let there be given (m) equations, which contain m unknown quantities $x, y, z, \&c.$; and let $\alpha, \beta, \gamma, \&c.$ be nearly correspondent values of the unknown quantities $x, y, \&c.$ respectively: assume $n+1$ different values of the quantity x , viz. $\alpha, \alpha + \pi, \alpha + \pi', \alpha + \pi'', \&c. \&c.$; and in like manner assume $n+1$ different correspondent values of the quantity y , which let be $\beta, \beta + \rho, \beta + \rho', \beta + \rho'', \&c.$; and so of the remaining; where $\pi, \pi', \pi'', \&c. \rho, \rho', \rho'', \&c. \&c.$ are very small quantities; substitute these quantities for their respective values in the given equations, and let the resulting quantities be $A, B, C, D, \&c.$ in the first equation; $P, Q, R, S, \&c.$ in the second, $\&c. \&c.$: assume from the first equation the n simple equations $a\pi + b\rho + \&c. = B - A, a\pi' + b\rho' + \&c. = C - A, a\pi'' + b\rho'' + \&c. = D - A, \&c.$; and from the second equation the n simple ones $b\pi + k\rho + \&c. = Q - P, b\pi' + k\rho' + \&c. = R - P, b\pi'' + k\rho'' + \&c. = S - P, \&c.$ From these equations can be investigated the co-efficients $a, b, \&c. h, k, \&c. \&c.$; ultimately assume the m equations $A + ae + bi + \&c. = 0, P + be + ki + \&c. = 0, \&c.$ from which can be deduced the values of the quantities $e, i, \&c.$; and $\alpha + e, \beta + i, \&c.$ will be more near values of the quantities, $x, y, \&c.$

11. Sir ISAAC NEWTON found the sum (A) of the $2n^{\text{th}}$ power of each of the roots of a given equation, and then extracted the $2n^{\text{th}}$ root of A , viz. $\sqrt[2n]{A}$ for an approximate value of the greatest root of the equation, and further added some similar rules on the same principle. 1a

In the *Miscell. Analyt.* and *Meditationes* the same principle is applied in different rules for finding approximates to the greatest and other roots of the given equation; and also limits of the ratios of the approximate values of the roots found by these rules to the roots themselves are given.

It is observed in the *Meditationes*, that from these rules in general to find the greatest root, it is often necessary that the greatest possible root be greater than the sum of the quantities contained in the possible and impossible part of any impossible root of the given equation: for example, if $a + b\sqrt{-1}$ be an impossible root of the given equation, then it is necessary that the greatest possible root be greater than $a + b$.

It may further be observed, that in equations of high dimensions (unless purposely made) it is probable, the number of impossible will greatly exceed the number of possible roots; and consequently these rules most commonly fail.

12. M. BERNOULLI assumed a fraction whose numerator is a rational function of the unknown quantity, and denominator the quantity, which constitutes the equation; and reduced the fraction into a series, whose terms proceed according to the dimensions of the unknown quantity; and thence found an approximate value of the greatest or least root of the given equation or its reciprocal, by dividing the co-efficient of any term of the series resulting by the co-efficient of the preceding or subsequent term: for example, let the equation be $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} \dots \pm Px^3 \mp Qx^2 \pm Rx \mp S = 0$; as-

$$\text{sume the fraction } \frac{nx^{n-1} - n-1px^{n-2} + n-2qx^{n-3} - n-3rx^{n-4} + \&c.}{x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c.} =$$

$$nx^{n-1} + (\alpha + \beta + \gamma + \delta + \&c.)x^{n-2} + (\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2 + \&c.)x^{n-3} + (\alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \epsilon^3 + \&c.)x^{n-4} + (\alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \epsilon^4 +$$

$\&c.)x^{-5} \dots + (\alpha^{m-1} + \beta^{m-1} + \gamma^{m-1} + \&c. = P)x^{-m} + (\alpha^m + \beta^m + \gamma^m + \&c. = Q)x^{-m-1} + (\alpha^{m+1} + \beta^{m+1} + \gamma^{m+1} + \&c. = R)x^{-m-2} + \&c.$; then will $\frac{R}{Q}$ or $\sqrt[m]{Q}$ be the greatest root nearly.

Ex. 2. $\frac{R - 2Qx + 3Px^2 - \&c. \dots nx^{n-1}}{S - Rx + Qx^2 - Px^3 + \&c. \dots x^n} = \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \&c. \right) + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} + \&c. \right)x + \left(\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \&c. \right)x^2 + \left(\frac{1}{\alpha^4} + \frac{1}{\beta^4} + \frac{1}{\gamma^4} + \&c. \right)x^3 + \dots + \left(\frac{1}{\alpha^{m-1}} + \frac{1}{\beta^{m-1}} + \frac{1}{\gamma^{m-1}} + \&c. = P \right)x^{m-2} + \left(\frac{1}{\alpha^m} + \frac{1}{\beta^m} + \frac{1}{\gamma^m} + \&c. = Q \right)x^{m-1} + \left(\frac{1}{\alpha^{m+1}} + \frac{1}{\beta^{m+1}} + \frac{1}{\gamma^{m+1}} + \&c. \right)x^m + \&c.$; then will $\frac{P}{Q}$ or $\sqrt[m]{\frac{P}{Q}}$, the least root nearly.

Ex. 3. $\frac{1}{x^n - px^{n-1} + qx^{n-2} - \&c.} = x^{-n} + (\alpha + \beta + \gamma + \delta + \&c.)x^{-n-1} + (\alpha^2 + \beta^2 + \gamma^2 + \&c. (+ \alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \&c.))x^{-n-2} + (\alpha^3 + \beta^3 + \gamma^3 + \&c. (+ \alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta + \&c.) + \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta + \&c.))x^{-n-3} + \&c.$; and $\frac{S}{S - Rx + Qx^2 - \&c. \dots x^n} = 1 + \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \&c. \right)x + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \&c. (+ \frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} + \&c.))x^2 + \&c.$ in each of which all the numeral co-efficients are 1. The approximate values to the greatest and least root may be found in the same manner as before.

From the preceding examples it appears, that the same observations which have been applied to Sir ISAAC NEWTON'S method are equally applicable to M. BERNOULLI'S.

13. In the *Meditationes* this rule is further extended, viz. let the given equation involve irrational and other functions of the unknown quantity; reduce it so that no function of the unknown quantity (x) may be contained in the denominator, and let the resulting equation be $A = 0$. Assume a fraction

$$\frac{B}{A}^2$$

$\frac{B}{A}$, whose numerator B is a finite rational and integral function of the unknown quantity; reduce $\frac{B}{A}$ into a series proceeding according to the dimensions of the unknown quantity: for example, let the series be $A'x' + B'x'^{+1} + Cx'^{+2} + \dots + Lx'^{+l} + Mx'^{+l+1} + Nx'^{+l+2} + \&c.$; then (*exceptis excipiendis*) if s be negative, will the greatest root be $\sqrt[l]{\frac{M}{L}}$ nearly; but, if s be affirmative, $\sqrt[l]{\frac{L}{M}}$ will be the least root nearly. If l be infinite, then (*exceptis excipiendis*, as before-mentioned) the quantities $\sqrt[l]{\frac{M}{L}}$ and $\sqrt[l]{\frac{L}{M}}$ will be the above-mentioned roots accurately.

These principles have been applied to find the remaining roots of the given equation as well as the greatest and least.

14. The rule of false has been found very useful in finding approximates to the two unknown quantities contained in two given equations, and has been applied to (n) equations having (n) different unknown quantities: for example, it has been observed, that if two or more (m) values of an unknown quantity (x) are nearly equal to each other and to its given approximate value (x') , the unknown quantity $v = x - x'$ will ascend to two or more (m) dimensions in one of the resulting equations; or in more than one equations will be contained such powers of the quantity (v) , that if the more equations were reduced to one whose unknown quantity is v , the resulting equation will contain (m) dimensions of the quantity v . Hence it appears, that in this case also the convergency of the approximate values found will depend on the given approximate being much more near to one root than to any other.

15. When

15. When the given equations $A=0$, $B=0$, $C=0$, &c. contain irrational or other quantities whose fluxions can be found; and approximates (a , b , c , d , &c.) are given to each of the unknown quantities (x , y , z , v , &c.) contained in the given equations; let $a+x'=x$, $b+y'=y$, $c+z'=z$, $d+v'=v$, &c.: substitute in the quantities $A, \left(\frac{A'}{x}\right), \left(\frac{A'}{y}\right), \left(\frac{A'}{z}\right), \left(\frac{A'}{v}\right)$, &c.; $\left(\frac{\ddot{A}}{x^2}\right), \left(\frac{\ddot{A}}{xy}\right), \left(\frac{\ddot{A}}{xz}\right), \left(\frac{\ddot{A}}{xv}\right)$, &c.; $\left(\frac{\ddot{A}}{xyz}\right), \left(\frac{\ddot{A}}{x^2y}\right)$, &c. for x, y, z, v , &c. respectively a, b, c, d , &c.; let the resulting correspondent values be $A', \left(\frac{A'}{x}\right), \left(\frac{A'}{y}\right)$, &c. whence may be deduced the equation $A' + \left(\left(\frac{A'}{x}\right)x' + \left(\frac{A'}{y}\right)y' + \left(\frac{A'}{z}\right)z' + \left(\frac{A'}{v}\right)v' + \&c.\right) + \left(\frac{1}{2}\left(\frac{A''}{x^2}\right)x'^2 + \frac{1}{2}\left(\frac{A''}{y^2}\right)y'^2 + \frac{1}{2}\left(\frac{A''}{z^2}\right)z'^2 + \&c. + \left(\frac{A'}{xy}\right)x'y' + \left(\frac{A''}{xz}\right)x'z' + \&c.\right) + \&c. = 0$ in which no irrational, &c. function of the approximates x', y', z' , &c. is contained; and in the same manner may the remaining equations $B=0$, $C=0$, &c. be transformed into others, in which no irrational function of the approximates (x', y', z' , &c.) is contained, and from the resulting equations may be found approximate values of the quantities x', y', z' , &c.

If there be given only two equations $A=0$ and $B=0$ containing two unknown quantities x and y , and all the quantities of the resulting equations, in which are contained more than one dimension of the quantities x' and y' be rejected, there will result the two equations $A' + \left(\frac{A'}{x}\right)x' + \left(\frac{A'}{y}\right)y' = 0$ and $B' + \left(\frac{B'}{x}\right)x' + \left(\frac{B'}{y}\right)y' = 0$, from which may be found x' and y' , the

same

same as given by Mr. SIMPSON and others. When two or more values of the quantity x are nearly $=a$, then in a resulting equation or equations, quantities of two or more dimensions of the approximate x' are to be included.

P A R T III.

1. THE first algebraists divided quantities, and extracted their roots no further than the quantities themselves: they did not perceive the utility of proceeding any further, otherwise the operation would have been the same continued. Mr. GREGORY ST. VINCENT, and Mr. MERCATOR divided, and Sir ISAAC NEWTON divided and extracted the roots of quantities (in which only one unknown quantity (x) is contained) by the operations then used by arithmeticians, into series ascending or descending, according to the dimensions of x in *infinitum*. They clearly saw the utility of it in finding the fluents of fluxions, as Dr. WALLIS and others some little time before had found the fluent of the fluxion $ax^{\frac{m}{n}}x$; or, which is the same, the area of a curve whose ordinate is $ax^{\frac{m}{n}}$ and abscissa is x .

2. M. LEIBNITZ asked from Mr. NEWTON the cases in which the above-mentioned serieses would converge; for it would be altogether useless when it diverges, and of little use when it converges slowly.

Caf. 1. To this question an answer, I believe, was first given in the *Meditationes*, viz. reduce the function to its lowest terms; and also in such a manner that the quantities contained in the numerator and denominator may have no denominator: make the denominator $Q=0$, and every distinct irrational quantity contained in it $=0$; and also every distinct irrational quantity H contained in the numerator $=0$; then, let α be the least root affirmative or negative (but not $=0$) of the above-mentioned resulting equations, the ascending series will always converge, if the value of x is contained between α and $-\alpha$; but if x be greater than α or $-\alpha$, the above-mentioned series will not converge.

If the above-mentioned series (S) be multiplied into \dot{x} , and its fluent found; then will the series denoting the fluent contained between two values a and b of the quantity (x) converge, when a and b are both contained between α and $-\alpha$: the fluent always converges faster than the series S, the unknown quantity x having the same values in both.

Ex. Let $\int \frac{\dot{x}}{a+bx+cx^2+\dots x^n} = Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \&c.$ and the roots of the equation $a+bx+cx^2+\dots x^n=0$ be $\alpha, \beta, \gamma, \delta, \&c.$; then $\frac{\dot{x}}{a+bx+cx^2+\dots x^n} = \frac{\pi\dot{x}}{\alpha-x} + \frac{\rho\dot{x}}{\beta-x} + \frac{\sigma\dot{x}}{\gamma-x} + \&c. = A\dot{x} + Bx\dot{x} + Cx^2\dot{x} + \&c.$; but $\frac{\pi}{\alpha-x} = \frac{\pi}{\alpha} + \frac{\pi x}{\alpha^2} + \frac{\pi x^2}{\alpha^3} + \&c.$ in infinitum, $\frac{\rho}{\beta-x} = \frac{\rho}{\beta} + \frac{\rho x}{\beta^2} + \frac{\rho x^2}{\beta^3} + \&c. \&c.$; the former series converges when x is contained between α and $-\alpha$, the latter when x is between β and $-\beta$, and so on. In the same manner the fluents $\int \frac{\pi\dot{x}}{\alpha-x} = \frac{\pi}{\alpha}x + \frac{\pi x^2}{2\alpha^2} + \&c.$ $\int \frac{\rho\dot{x}}{\beta-x} = \frac{\rho}{\beta}x + \frac{\rho x^2}{2\beta^2} + \&c. \&c.$ *a fortiori* converge when x is between α and $-\alpha$, β and $-\beta$, $\&c.$ respectively, and so on: hence the series $Ax + \frac{1}{2}Bx^2 + \frac{1}{3}Cx^3 + \&c.$

4

$= \int$

$$= \int \frac{x}{a+bx+cx^2 \dots x^n} = \int \frac{\pi x}{\alpha-x} + \int \frac{e x}{\beta-x} + \int \frac{\sigma x}{\gamma-x} + \&c. = \left(\frac{\pi}{\alpha} + \frac{e}{\beta} + \frac{\sigma}{\gamma} + \&c. \right) x + \left(\frac{\pi}{\alpha^2} + \frac{e}{\beta^2} + \frac{\sigma}{\gamma^2} + \&c. \right) x^2 + \&c.$$
 always converges when x is between α and $-\alpha$, where α is the least root of the above-mentioned equation; but where x is greater than α or $-\alpha$, the series will diverge.

The infinite series $a^m + ma^{m-1}x + m \cdot \frac{m-1}{2} a^{m-2}x^2 + \&c. = \overline{a+x}^m$ will always converge when a is greater than x , and diverge when less, and consequently its convergency does not depend on the index m , unless when $x = \pm a$: and in the *Meditationes Analyticae* are given the cases in which it converges or diverges when $\mp a = x$; and also if the series $x^m + max^{m-1} + \&c. = \overline{x+a}^m$ descends according to the dimensions of x , when it converges or diverges.

Caf. 2. Let the above-mentioned quantity be reduced into a series $Ax^{-r} + Bx^{-r-1} + \&c.$ descending according to the dimensions of the unknown quantity x ; then will the series $Ax^{-r} + Bx^{-r-1} + \&c. = P$, or the series $\frac{Ax^{-r+1}}{1-r} - \frac{Bx^{-r}}{r} + \&c. = \int Px$ converge, when x is greater than the greatest root (λ) of the above-mentioned equations, and diverge when it is less; and consequently in this case, when the fluent is required between the two values a and b of x ; the series found will converge when a and b are both greater than λ .

Caf. 3. When x is equal to the least root in the former case, and to the greatest in the latter, then sometimes the series will converge, and sometimes not. These different cases are given in the *Meditationes*; but it would be too long to insert them in this Paper.

4. If any roots are impossible as $a - b\sqrt{-1}$ and $a + b\sqrt{-1}$, then the series will converge when x is in the first case

less than $\pm(a-b)$, and in the second case greater than $\pm(a+b)$; or, more general, it will converge in the first case when x^n is always infinitely less than the reciprocal of the quantity $\frac{(a+b\sqrt{-1})^n + (a-b\sqrt{-1})^n}{(a^2+b^2)^n} = P$, where n is infinite; and in the latter case it will converge when x^n is infinitely greater than P .

It may not be improper to observe, that the same values of the root are to be applied in the equations, which are applied in the series.

3. Sir ISAAC NEWTON, in the binomial theorem, reduced the power or root of a binomial into a series proceeding according to the dimensions of the terms contained in the binomial. M. DE MOIVRE reduced the power or root of a multinomial into a like series; but in all cases, except the most simple, we must still recur to the common division, extraction of roots, &c.

4. Mess. EULER, MACLAURIN, and other mathematicians, finding that the serieses before-mentioned often converged slowly, or, if the truth may be confessed, commonly not at all, to deduce the area of a curve contained between two values a and b of the absciss, or fluent of a fluxion between two values a and b of the variable quantity x , interpolated the series or area between a and b ; that is, found the area or fluent contained between the abscissæ a and $a+\alpha$, then between the abscissæ $a+\alpha$ and $a+2\alpha$, and then between the abscissæ $a+2\alpha$ and $a+3\alpha$, and so on, till they came to the area between $b-\alpha$ and b . M. EULER observed, that when the ordinate became 0 or infinite, the series expressing the area converges slowly; and therefore,

in order to investigate the area near the points of the absciss, where the ordinates become 0 or infinite, he transforms the equation, and finds serieses expressing the area near those points, in which serieses the abscissæ or unknown quantities begin from those points.

5. In the *Meditationes* it is asserted, that in a series proceeding according to the dimensions of x , if any root of the above-mentioned equations be situated between the beginning of the absciss 0 and its end x , the series will not converge: it is therefore necessary to transform the absciss so that it may begin or end at each of the roots of the above-mentioned equations, and consequently where the ordinates become 0 or infinite, &c.; those cases excepted where the ordinate becomes 0, and its correspondent abscissa is a root of a rational function W of x without a denominator, and $\int WPx$ is equal to the given series; and in general the abscissæ ought to begin from the above-mentioned points; for if they end there the series will converge very slow, if at all.

6. It is further asserted, that if a and b , the values of the abscissæ between which the area is required, be both more near to one root of the above-mentioned equations than to any other, and n serieses are to be found, whose sum expresses the area contained between a and b ; then that the sum of the (n) serieses may converge the swiftest, the distances of the beginnings of each of the n abscissæ from the adjacent root will form a geometrical progression.

7. Mr. CRAIG found the fluent of any fluxion of the formula $(a + bx^n + cx^{2n} + \&c.)^{n x^0 - 1} x$ by a series of the following

kind $(a + bx^n + cx^{2n} + \&c.)^{m+1} \times x^0 \times (\alpha + \beta x^n + \gamma x^{2n} + \&c. \text{ in infinitum})$. Sir ISAAC NEWTON, by serieses of the same kind, found the fluents of fluxions of this formula $(a + bx^n + cx^{2n} + \&c.)^l \times (e + fx^n + gx^{2n} + \&c.)^m \times \&c. x^{0-1} \dot{x}$; the same principle is extended somewhat more general in the *Meditationes*.

8. Mr. JOHN BERNOULLI found the fluent of any fluxion

$$\int n \dot{x} = nx - \frac{x^{2n}}{2n} + \frac{x^{3n}}{2 \cdot 3n^2} - \&c. \text{ from the principles which Mr.}$$

CRAIG published for finding the fluents of fluxions involving fluents.

In the *Meditationes* something is added of the convergency of these series; and also,

9. In them a new method is given of finding approximations. Let some terms in the given quantity be much less or greater than the rest; then reduce the quantity into terms proceeding according to the dimensions of the small quantities, or according to the reciprocals of the great quantities, and it is done. If the fluent of the quantity resulting is required, find it from the common methods, if possible; but if not, reduce the terms not to be found into an infinite series, and then find approximate values to each of the terms, &c.

Ex. 1. Let R the radius, and A the arc of a circle whose sine is S and cosine C, and $A \pm e$ an arc of a circle which does not much differ from the arc A, that is, let e be a very small quantity: then will the sine of the arc $A \pm e$ be $S(1 - \frac{1}{2} \times \frac{e^2}{R^2} + \frac{1}{2 \cdot 3 \cdot 4} \times \frac{e^4}{R^4} - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \times \frac{e^6}{R^6} + \&c.) \pm C(\frac{e}{R} - \frac{1}{2 \cdot 3} \times \frac{e^3}{R^3} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} \times \frac{e^5}{R^5} - \&c.)$; and the cosine of the same arc will be $C(1 - \frac{1}{2} \times \frac{e^2}{R^2} + \frac{1}{2 \cdot 3 \cdot 4} \times \frac{e^4}{R^4} - \&c. \text{ in infinitum}) \pm S(\frac{e}{R} -$

$$\frac{1}{2 \cdot 3} \times \frac{e^3}{R^3} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} \times \frac{e^5}{R^5} - \&c.).$$

Ex. 2. Log. $(x \pm e) = \int \frac{\dot{x}}{x \pm e} = \log. x \pm \frac{e}{x} \mp \frac{e^2}{2x^2} \pm \frac{e^3}{3x^3} \mp \&c.:$

$$\int \frac{\dot{x}}{a^2 - (x+e)^2} = \int \frac{\dot{x}}{a^2 - x^2} + \frac{e}{a^2 - x^2} + \&c.$$

Ex. 3. $\int \frac{\dot{x}}{a^2 + (x-e)^2} = \int \frac{\dot{x}}{a^2 + x^2} - \frac{e}{a^2 + x^2} + \&c.$

Ex. 4. $\int \frac{\dot{y}}{\sqrt{(1-y \pm e)^2}} = \int \frac{\dot{y}}{\sqrt{1-y^2}} \pm \frac{e}{\sqrt{1-y^2}} \pm \&c.$

Ex. 5. $\int \frac{\dot{y}}{\sqrt{1-y^2} + e} = \int \frac{\dot{y}}{\sqrt{1-y^2}} - \int \frac{e\dot{y}}{1-y^2} + \int \frac{e^2\dot{y}}{(1-y^2)^{3/2}} - \int \frac{e^3\dot{y}}{(1-y^2)^2} + \&c.$

Ex. 6. Let the fluxion be $\frac{\sqrt{1-cx^2}\dot{x}}{\sqrt{1-x^2}}$, where c is a very small

quantity; then, if P be put for $\sqrt{1-x^2}$, the fluxion becomes

$$\frac{\dot{x}}{P} - \frac{cx^2\dot{x}}{2P} - \frac{c^2x^4\dot{x}}{8P} - \frac{c^3x^6\dot{x}}{16P} - \&c. \text{ of which the fluents will be found}$$

$$A - \frac{c}{2} \times \frac{1 \cdot A - xP}{2} - \frac{c^2}{8} \times \frac{3B - x^3P}{4} - \frac{c^3}{16} \times \frac{5C - x^5P}{6} - \&c. \text{ where } A =$$

$$\int \frac{\dot{x}}{P}, B = \frac{1 \cdot A - xP}{2}, C = \frac{3B - x^3P}{4}, \&c.$$

This is the swiftest converging series for finding the length of the arc of an ellipse nearly circular, which is yet known; for example, let the absciss to the axis beginning from the center = x , the semi-transverse axis of the ellipse be 1, its semi-conjugate $1-d$; then will $c = 2d-d^2$, and let the length of the quadrant of the ellipse be required, in this case $x=1$, and $P = \sqrt{1-x^2} = 0$; and $A = \frac{3,14159, \&c.}{2} = 1,57079,$

$\&c.$ whence the length required is $1,57079, \&c. \times (1 - \frac{c}{2 \cdot 2} -$

$$\frac{1 \cdot 3 c^2}{2^2 \cdot 4^2} - \frac{1 \cdot 3^2 \cdot 5 c^3}{2^2 \cdot 4^2 \cdot 6^2} - \frac{1 \cdot 3^2 \cdot 5^2 \cdot 7 c^4}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \&c.).$$

Ex.7. Let the fluxion be $\dot{x}(a^3 + a' + (3a^2b + b')x + (3ab^2 + c')x^2 + (b^3 + d')x^3)^m = P\dot{x}$, where $a', b', c', \&c.$ are very small quantities; then will $P\dot{x} = ((a + bx)^{3m} + m\overline{a + bx}^{3m-3} \times (a' + b'x + c'x^2 + d'x^3) + m \cdot \frac{m-1}{2} \overline{a + bx}^{3m-6} \times (a' + b'x + c'x^2 + d'x^3)^2 + \&c.) \dot{x}$,

of which the fluent is $\frac{1}{3m+1 \cdot b} \overline{a + bx}^{3m+1} + m(a + bx)^{3m-2} \times \left(\frac{d'}{(3m+1)b} x^3 + \frac{c' - 3aA}{3mb} x^2 + \frac{b' - 2aB}{(3m-1)b} x + \frac{a' - aC}{(3m-2)b} \right) + \&c.$ where the letters A, B, C, &c. denote the preceding co-efficients.

10. M. EULER and others, reduced the series $Ax^r + Bx^{r+1} + Cx^{r+2} + \&c.$ into a series $A' \sin. ra + B \sin. r + sa + \&c. \&c.$ where α denotes the arc of a circle, whose sine is ax , &c. It may be easily reduced into infinite other serieses proceeding according to the dimensions of quantities, which are functions of x ; but it is most commonly preferable to reduce it into serieses proceeding according to the sines, cosines, tangents, or secants of the arcs of circle, which sines, &c. can immediately be procured from the common tables.

It has been observed in the first part, that to find the root of an equation, an approximate value much more near to one root of the equation than to any other must be given. In this part it is further observed, that serieses deduced from expanding given quantities, so as to proceed according to the dimensions of the unknown or variable quantities, will not converge if the unknown quantities be greater than the least roots of the above-mentioned equations; and that they will not converge much, unless the unknown quantities have a small proportion to the least roots: and if the given quantities be expanded into serieses descending according to the dimensions of the unknown quantities, then the serieses resulting will not converge if the
greatest

greatest roots of the equations before-mentioned be greater than the unknown quantities; and unless the unknown quantities have a great ratio to the greatest roots the serieses will converge slowly: for example, the serieses $\int \frac{x}{1+x} = x - \frac{1}{2}x^2 + \&c.$, $\int \frac{z}{1+z^2} = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \&c.$, $\int \frac{y}{\sqrt{1-y^2}} = y + \frac{1}{6}y^3 + \&c.$ will never converge if x , z , or y , be greater than 1; but will always converge when less than ± 1 or $\pm 1 \sqrt{-1}$ the least or only roots of the equations $1+x=0$, $1-y^2=0$, and $1+z^2=0$. The series $y + \frac{1}{6}y^3 + \&c.$ will always converge when y is situated between $+1$ and -1 , in which case alone it is possible. The same is true also of a series arising from expanding the $\int (ax^m + bx^{m-1} + cx^{m-2} + \&c.)^{\frac{1}{m}} dx$ into a series proceeding according to the dimensions of x , if the equation $ax^m + bx^{m-1} + cx^{m-2} + \&c. = 0$ has only two possible roots α and $-\alpha$, which are less in the manner before-mentioned than any impossible root contained in it.

If in either of the above-mentioned serieses the unknown quantity x , z , or y , has a great proportion to 1, the series will converge very slow; for example, if $x=1$, ten thousand numbers at least are to be calculated, to procure the sum of the series true to four figures; therefore, in these and most other serieses it is necessary first to find a near value, viz. when x either $=z$, when e is very small; or $=e$, when z is very small; and then write $z+e$ for x in the quantity, and reduce it in the former case into a series proceeding according to the dimensions of e , in the latter case according to the dimensions of z , and there will arise two serieses, of which the fluents properly corrected, viz. by adding the fluent contained between the values a and e to the latter, and that between a and z to the former, will give the same
 fluent.

fluent. The first term of the series, in which e is supposed very small, will be the fluent of the given fluxion, when $x = z$.

11. If a fluxion $P\dot{x}$, where P is a function of x , be transformed into another $Q\dot{z}$, where Q is a function of z , and they be reduced into serieses A and B , proceeding according to the dimensions of x and z respectively; find α and π , correspondent values of the quantities x and z ; then in ascending serieses, if α has a less ratio to the least root of the equation $P=0$, than π has to the least root of the equation $Q=0$, the series A (*exceptis excipiendis*) will converge swifter than the series B .

12. Dr. BARROW, in some equations, expressing the relation between the absciss x and ordinate y , found y in the two first terms of x , viz. $y = a + bx$, which is an equation to the asymptotes of the curves. Sir ISAAC NEWTON, from an algebraical equation given, expressing the relation between y and x , found a series proceeding according to the dimensions of x , expressing y in terms of x . M. LEIBNITZ performed the same problem by assuming a series $Ax^n + Bx^{n+r} + Cx^{n+2r} + \&c.$ with general co-efficients, and substituting this series for y in the given equation, &c. from equating the correspondent terms he deduced the indexes and co-efficients. M. DE MOIVRE, Mr. MAC LAURIN, &c. observed, that when the highest terms of the given equations have two or more (m) divisors equal; for example, $(y - ax^n)^m$; to which we must add, and when a value of y in this case is required nearly equal to Ax^n , a series $Ax^n + Bx^{n+\frac{r}{m}} + \&c.$ is to be assumed, whose indexes differ only by $\frac{r}{m}$, &c. if otherwise they would differ by r .

This reduction seldom answers any other purpose than finding the fluents of fluxions as $\int y\dot{x}$, &c.; or the asymptotes, &c.

of

of curves, which depend on some of the first terms of the series; but will very seldom be used for finding the roots of an equation; the rule of false, or method given by VIETA, will ever be substituted in its stead.

13. The values of x may be required between which the above-mentioned series $Ax^n + Bx^{n+r} + Cx^{n+2r} + \&c.$ will converge, as the infinite series answers no purpose when it diverges. First, if an ascending be required, write for y in the given algebraical equation an infinite quantity, and find the roots of x in the equation thence resulting $P = 0$; which for y write in the same equation, and find the roots of x in the resulting equation which contain irrational quantities, *viz.* if one root be $x = a$; then let it contain $(x - a)^m$, where m is not a whole number; find the roots of the equations resulting from making every irrational function of (x) contained in the given equation $= 0$, there being no irrational function of y contained in it; then, if x be greater than the least root not $= 0$ of the above-mentioned equations, the series will not converge; but if it be a series descending according to the dimensions of x , and x be less than the greatest root of the above-mentioned equations, the series will not converge.

In interpolating serieses to investigate the fluent contained between two values a and b of the fluxion $(Ax^n + Bx^{n+r} + \&c.)t\dot{x}$, it is preferable to make the abscissæ begin from every one of the above-mentioned roots contained between a and b .

Most commonly these serieses will not converge unless x be less, &c. than other quantities not investigated by this rule.

14. Sir ISAAC NEWTON gave an elegant example of this rule in the reversion of the series, $y = ax + bx^2 + cx^3 + \&c.$ from which the investigation of the law of the series has never been attempted. In the year 1757 I sent the first edition of my

Meditationes Algebraicæ to the Royal Society, and published it in 1760, and afterwards in 1762, with another part added, on the Properties of curves, under the title of *Miscellanea Analytica*, in all which was given the law of a series for finding the sum of the powers of the roots of an equation from its co-efficients. That great mathematician M. LE GRANGE and myself printed about the same time an observation known to me at the time that I printed the above-mentioned book, that the law of its powers and roots, if it proceeds *in infinitum*, is the same; from which series of mine, with great sagacity, M. LE GRANGE found the law which Sir ISAAC NEWTON's reversion of series observes. In the *Meditationes* the law is given, and the series is made to proceed according to the dimensions of x , &c.

15. It is asserted in the *Meditationes*, that in most equations of high dimensions, unless purposely constituted, the sum of the terms which, from the given hypothesis, become the greatest, being supposed $= 0$, only an approximate to the value Ax^n of y in the resulting equation can by the common algebra be deduced. In this case the approximate to the quantity A is to be found so near as the approximate value of the quantity sought requires; or perhaps it is preferable to correct in every operation the approximate values of the quantities A, B, C , &c. in the series required $A'x^n + B'x^{n+r} + C'x^{n+2r} + \&c.$

In the equation the quantity $z \pm e$ may be substituted for x , and from the equation resulting a series expressing the value of y may be found, proceeding either according to the dimensions of the quantity z , or its reciprocals, according to the conditions of the problem.

16. If there are more than one (n) equations having ($n+1$) unknown quantities (x, y, z , &c.): in each of the equations for y ,

z , &c. write respectively Ax^n , $A'x^m$, &c.; and suppose the terms of each of the equations, which result the greatest from the given or assumed hypothesis $= 0$, and from the resulting equations may be found the first approximates Ax^n , $A'x^m$, &c. either accurately or nearly; then, in the given equations for y , z , &c. write $y' + (A + a)x^n + Bx^{n+n'} + \&c. z' + (A' + a')x^m + B'x^{m+m'}$, where a , a' , &c. are very small quantities; and suppose the terms of each of the equations which become greatest from the above-mentioned hypothesis respectively $= 0$, and from the equations resulting deduce the quantities a , a' , &c.; n' , m' , &c.; B , B' , &c.; and so on: or assume $y = (A + 1a + a1 + \&c.)x^n + (B + 1b + b1 + \&c.)x^{n+n1} + \&c.$; $z = (A' + 1a' + a'1 + \&c.)x^m + (B' + 1b' + b'1 + \&c.)x^{m+m1} + \&c.$; substitute these quantities for their values in the given equations, and from equating the correspondent terms of the resulting equations may be deduced the quantities required.

The differences of the indexes n' , &c. m' , &c. may be deduced by writing x^n , x^m , &c. for y , z , &c. in the given equations, from the differences of the indexes of the quantities resulting. The same principles may be applied in finding the above-mentioned differences, when two or more values are Ax^n , &c. which were applied in a like case to one equation having two unknown quantities.

The same principles which discover the cases in which a series deduced from an equation having two unknown quantities will converge, may be applied for the same purpose to these series.

17. In the equations for x , y , z , &c. write respectively $x' + e$, $y' + f$, $z' + g$, &c.; and from the equations resulting find y' , z' , &c. in serieses either proceeding according to the dimensions

of the quantities $e, f, g, \&c.$; or according to the dimensions of the quantity x' , as the conditions of the problem require.

18. Given one or more (n) algebraical equations involving ($n+m$) unknown quantities, one unknown quantity (y) may be expressed by a series proceeding according to the dimensions of the $m-1$ remaining ones ($x, z, v, \&c.$), in which any dimensions of $z, v, \&c.$ assumed may be supposed to correspond to (l) dimensions of the quantity (x).

19. In a fluxional equation of the m^{th} order, expressing the relation between x, y , and their fluxions, where \dot{x} is constant, Mr. EULER substitutes in the given equation for $y^m, y^{m-1}, y^{m-2}, y^{m-3}, \&c.$ respectively $Ax^{n-m}\dot{x}^m, \frac{A}{n-m+1}x^{n-m+1}\dot{x}^{m-1} + a\dot{x}^{m-1}, \frac{A}{(n-m+1)(n-m+2)}x^{n-m+2}\dot{x}^{m-2} + ax\dot{x}^{m-2} + b\dot{x}^{m-2}, \frac{A}{(n-m+1)(n-m+2)}x^{n-m+3}\dot{x}^{m-3} + \frac{1}{2}ax^2\dot{x}^{m-3} + bx\dot{x}^{m-3} + c\dot{x}^{m-3}, \&c.$ where $a, b, c, \&c.$ are any quantities to be assumed in such a manner as the conditions of the problem require; from supposing the aggregate of the terms of the resulting equation, which are the greatest, $=0$, may be deduced the first approximate Ax^n , or else (as is beforementioned) $A'x^n$ a near approximate to Ax^n , and by proceeding as in algebraical equations another approximate may be found, and so on. The same may be found by assuming $y = Ax^n + Bx^{n+r} + Cx^{n+2r} + \&c. + ax^m + bx^{m-1} + cx^{m-2} + \&c.$ or $y = (A + 1a + a1 + \&c.)x^n + (B + 1b + b1 + \&c.)x^{n+r} + (C + 1c + c1 + \&c.)x^{n+2r} + \&c. + ax^m + bx^{m-1} + cx^{m-2} + \&c.$ and substituting it and its fluxions for their values $y, \dot{y}, \ddot{y}, \&c.$ in the given equation, and supposing the aggregate of each correspondent terms, which do not very much differ from each other, $=0$; from the resulting equations can be deduced the co-efficients $A, B, C, \&c.$; or $A, 1a, a1, \&c.$; $B, 1b, b1, \&c.$; $C, 1c, c1, \&c. \&c.$

In the given equation for y , x , and their fluxions substitute $y' + f$, $x' + g$, and their fluxions, where the quantities f and g , &c. are assumed, so as to render the quantities y' and x' , &c. very small.

In finding the series which expresses the value of y in terms of x , there will always occur as many invariable quantities to be assumed at will as is the order of the fluxional equation, provided the series begins from its first terms; and to find them there will result equations easily reducible to homogeneous fluxional equations, of which the orders do not exceed m .

